Midterm Computational Methods of Science/Computational Mechanics, December 2021

Duration: 2 hours.
In front of the questions, one finds the points. The sum of the points plus 1 gives the end mark for this test. Criteria used for the grading are: demonstration of understanding, logical reasoning, correct use of terminology, correctness of results.
The indication "Math" means for math students, and similar "ME" for mechanical engineering students.

Consider on (0,1) the differential equation

$$-\frac{d}{dx}(\cos(x)\frac{du}{dx}) + \frac{1}{1+x^2}u = \tan(x)$$

with boundary conditions $u(0) = -3$ and $\frac{du}{dx}(1) + u(1) = 2$.

1. [2.5] Derive the weak (Galerkin) form and the associated function space of this problem. Give also the bilinear and linear form, where in the bilinear form only first derivatives appear.

2. [1.0] Choose the remaining freedom in the bilinear form such that it becomes nonnegative and show that it is actually positive definite then.

3. [1.0] Show that the bilinear form is also symmetric and deduce from that that the matrix will become symmetric if we approximate the problem on a subspace with basis functions $\phi_1(x), \cdots, \phi_n(x)$. Also show that the matrix will be positive definite.

4. Math[1.0] Formulate the Lax-Milgram theorem and use it to show that the weak form is well-posed according to the Lax-Milgram theorem? Hint: observe that $u(1) = u(0) + \int_0^1 \frac{du}{dx}dx$, where the last term is the innerproduct of du/dx and the constant function 1.

5. [2.5] Given the interpolation points $x_i = ih$, with $h = 1/n$ on which we define the space of piecewise linear interpolation polynomials V_h in which each element u_h has the property $u_h(0) = 0$. Using the linear basis functions, give the expression for a diagonal element of the matrix from the weak problem on this subspace occurring for $i = 1, \cdots, n-1$ and the one for $i = n$. It is enough to express the diagonal element in integrals which you don't have to determine.

6. [0.5] Give the associated minimization problem. Also here, express your final results in integrals.

7. [0.5] Suppose we use instead of piecewise linear polynomial interpolation piecewise quadratic polynomial interpolation. What will be the expected order of convergence of $||u - u_h||$ and of $||u - u_h||_{H_1}$, respectively? Here u is the exact solution and u_h the solution on the subspace. Motivate your answer.

8. ME [1.0] In elasticity problems, what is meant by rigid motions? What kind of problem occurs in the solution process if the weak form of an elasticity problem allows rigid motions? How to cure this?